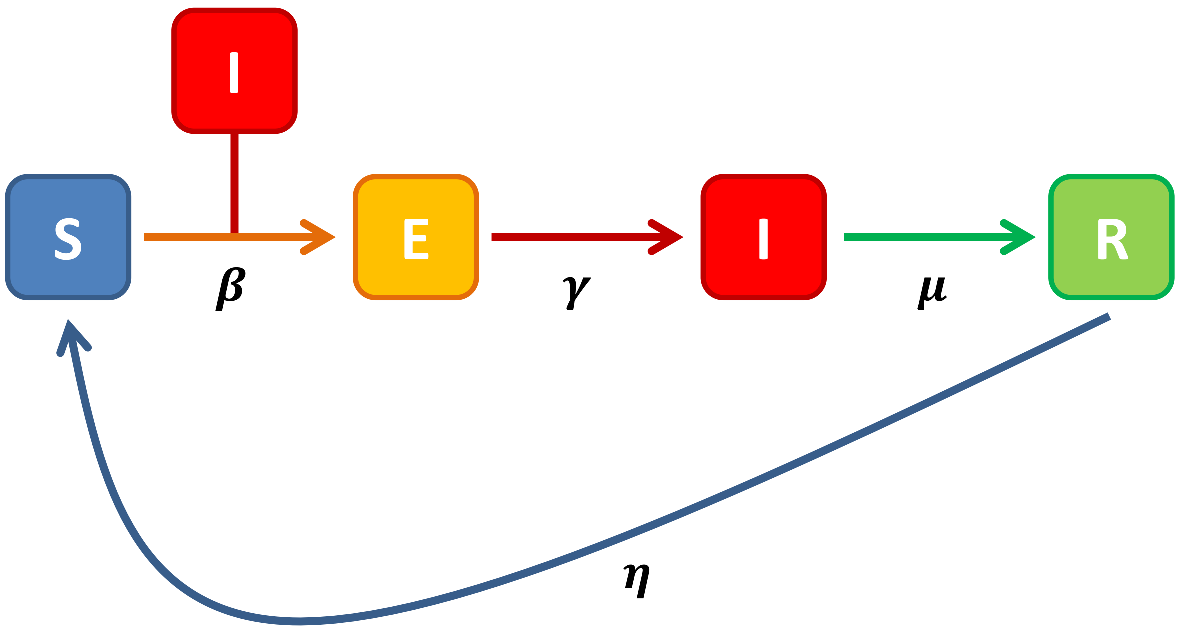
Mean field theory

Effective mean field (law of mass action)

Ignoring (many) correlations

Epidemic spreading on networks



Reference: Pastor-Satorras *et al.*, arXiv:1408.2701

(Continuous-time) Markov Chain

Exact ☺

Hard ☹

Example: Markov chain for SIS on a network

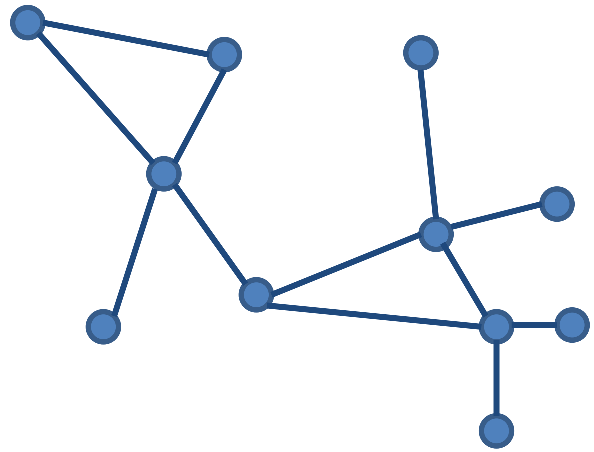
N nodes
2 states per node {S,I} ({0,1})

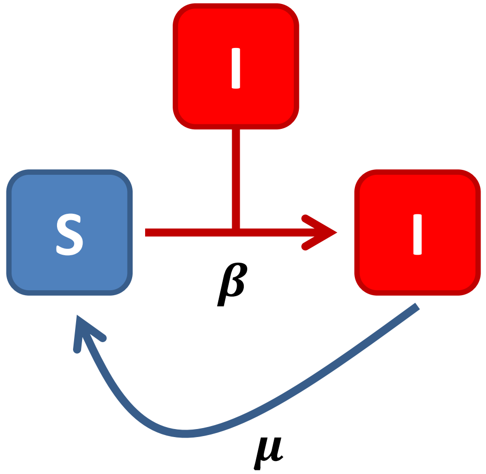
System state: ex. {S,I,S,S,…,I,S}

2^N states for the system
(2^Ncoupled equations).

Propagator is $2^N \times 2^N$ matrix.

Unfeasible for large systems



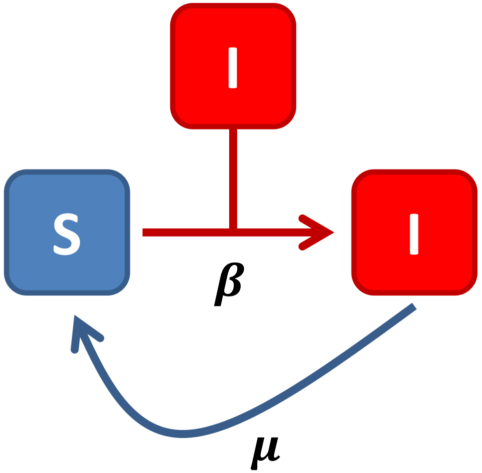


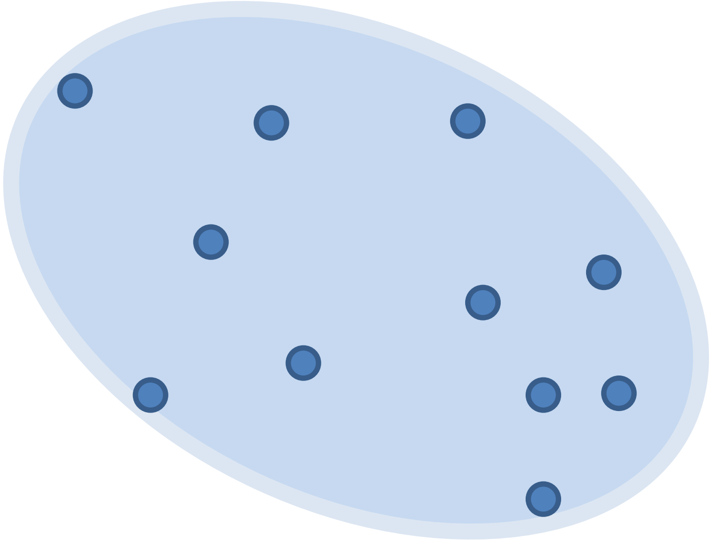
(Homogeneous) Mean Field (MF)

Simple ☺

No network ☹

Mean field (MF): SIR process

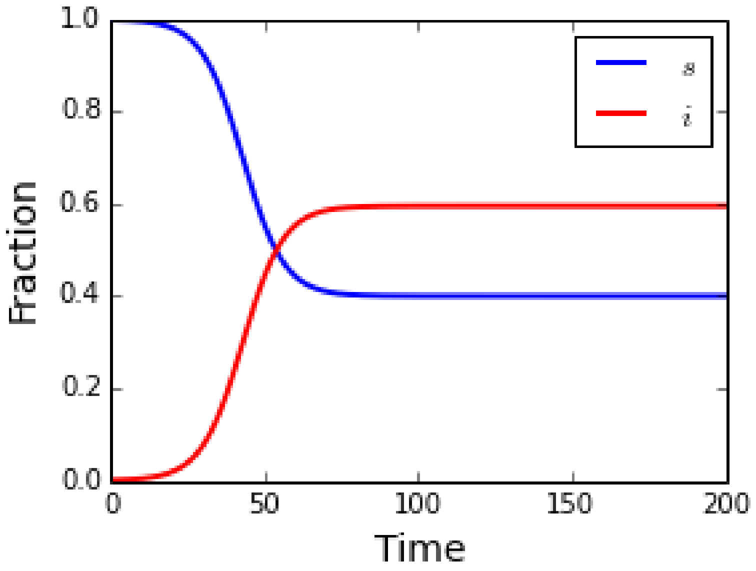




2 equations:

$$\frac{ds}{dt} = -\beta\langle k\rangle si + \mu i$$

$$\frac{di}{dt} = \beta\langle k\rangle si - \mu i$$



MF: SIS—Epidemic size and threshold

Use $s + i = 1$ to get 1 equation:

$$\frac{di}{dt} = \beta\langle k\rangle(1 - i)i - \mu i$$

Short-term behavior:

Linearize to find epidemic threshold:

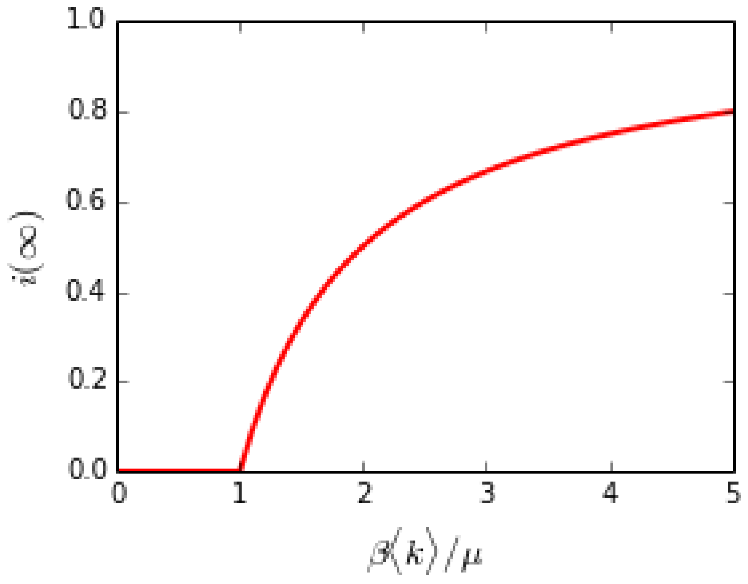
$$\frac{di}{dt} \approx \beta\langle k\rangle i - \mu i$$

$$\beta_c\langle k\rangle = \mu$$

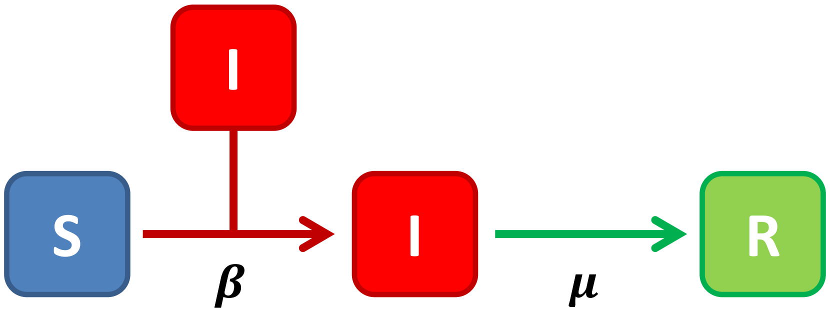
Long-term behavior:

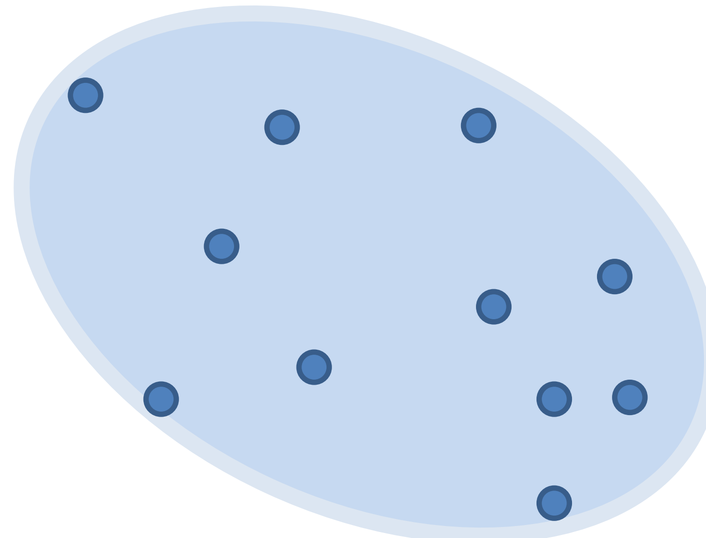
$$0 = \beta\langle k\rangle(1 - i)i - \mu i$$

$$\mathrm{i}(\infty) = \max(\mu = 0, \mu = 1 - \mu/\beta\langle k\rangle)$$



Mean field (MF): SIR process



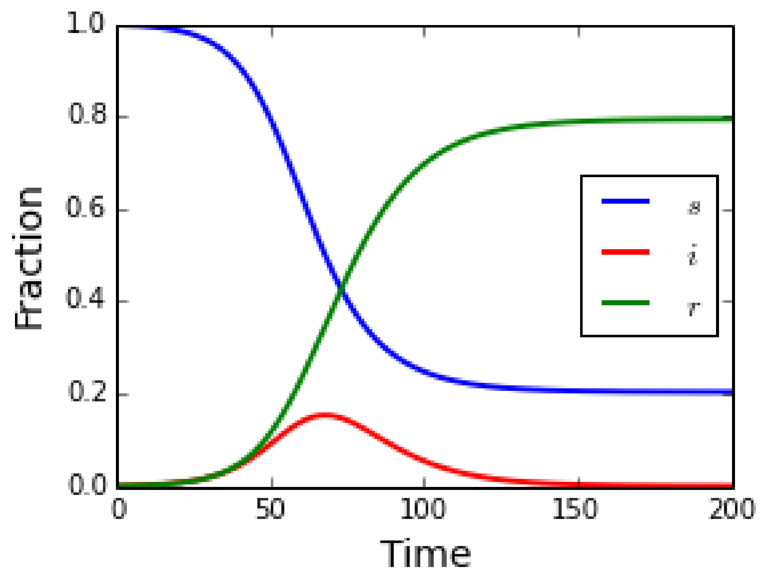


3 equations:

$$\frac{ds}{dt} = -\beta\langle k\rangle si$$

$$\frac{di}{dt} = \beta\langle k\rangle si - \mu i$$

$$\frac{dr}{dt} = \mu \mathrm{i}$$



MF: SIR—Epidemic size and threshold

Integrate to find $s(\infty)$ and $r(\infty)$:

$$s(\infty) = \exp\left(-\beta\langle k\rangle \int_0^\infty i(t)dt\right)$$

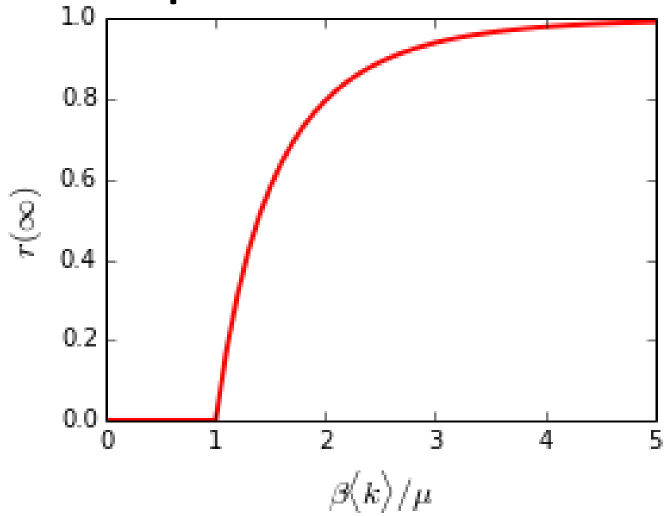
$$r(\infty) = \mu \int_0^\infty i(t)dt$$

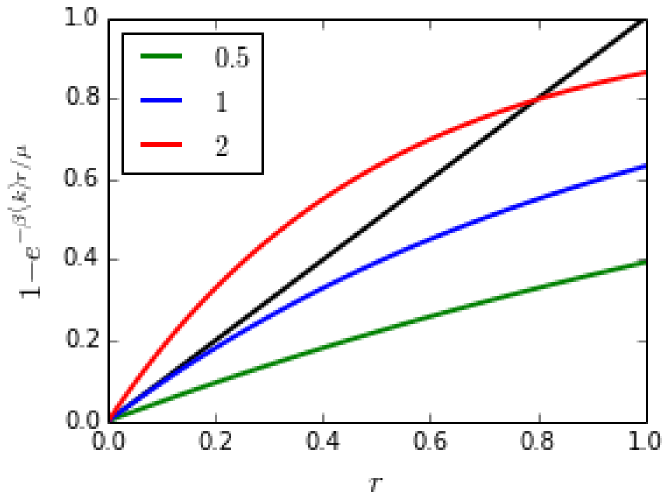
Use $r(\infty) + s(\infty) = 1$:

$$r + \exp\left(-\frac{\beta\langle k\rangle r}{\mu}\right) = 1$$

Threshold: $\beta_c\langle k\rangle = \mu$

Epidemic size



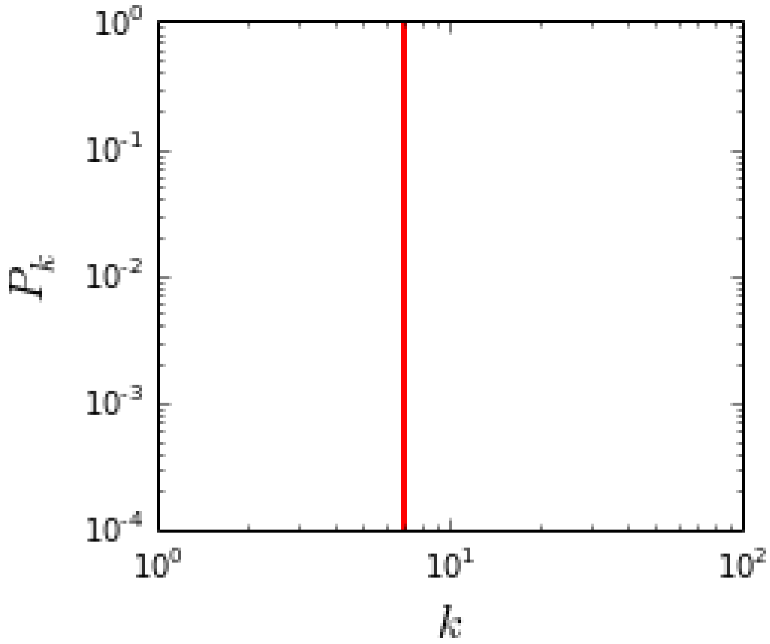


MF: The downside

Crucial assumption is wrong:

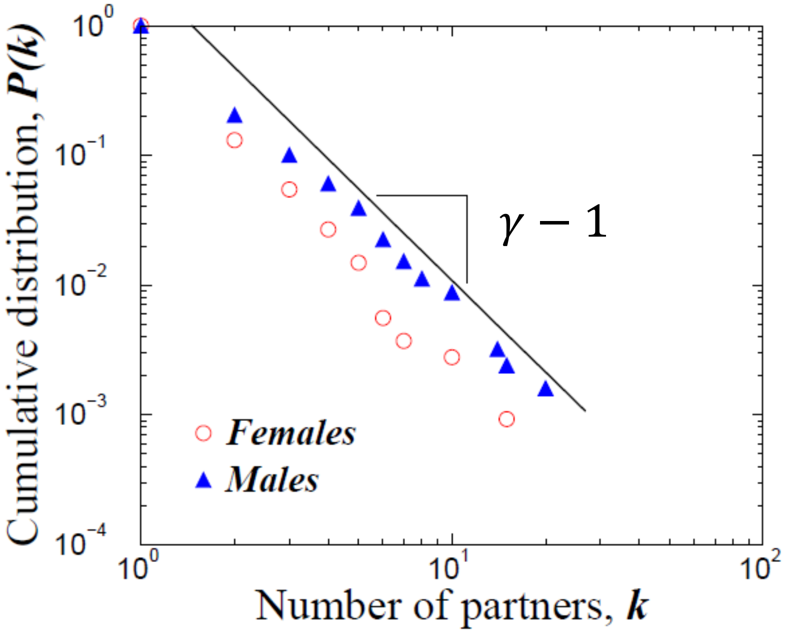
Homogeneous mixing,

$$P_k = \delta_{k,\langle k \rangle}$$



Heterogeneous mixing,

$$P_k \sim k^{-\gamma}$$



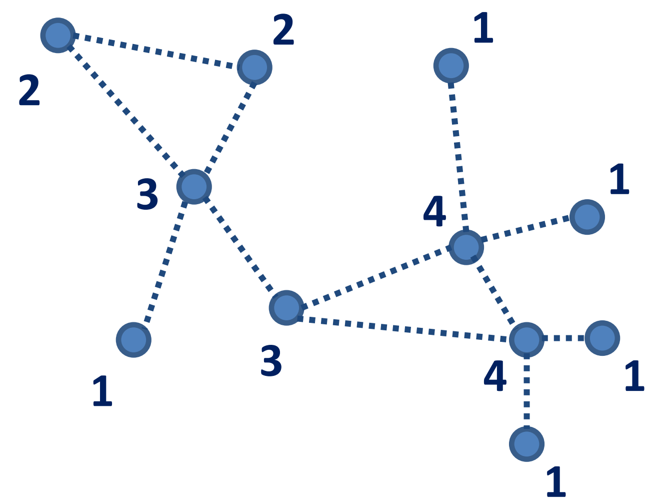
Degree-based Mean Field (DBMF)
Heterogeneous Mean Field (HMF)

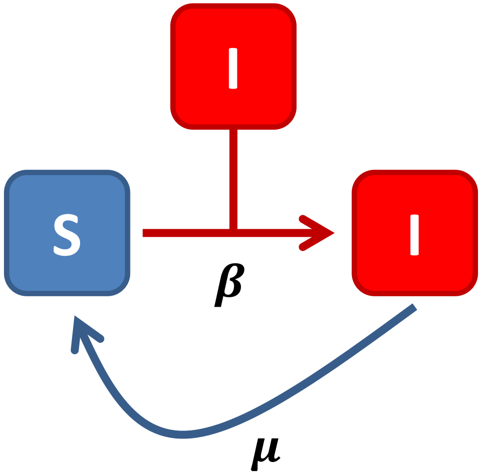
Annealed network 😐

Degree-based mean field (DBMF)

Keep degree distribution P_k

(Maybe) degree correlations $P_{k'|k}$





$(k_{\max}+1)$ **equations:**

$$\frac{di_k}{dt} = -\mu i_k + \beta k(1 - i_k)\Theta$$

$$\Theta = \sum_{k'} P_{k'|k}\, i_{k'}$$

DBMF: SIS—Epidemic threshold

$$\frac{di_k}{dt} = -\mu i_k + \beta k(1 - i_k)\Theta$$

Assuming uncorrelated degrees:

$$\Theta = \sum_{k'} \frac{k' P_{k'}}{\langle k \rangle} i_{k'}$$

Summing over k:

$$\frac{d\Theta}{dt} = -\mu\Theta + \beta\Theta \sum \frac{k^2 P_k (1 - i_k)}{\langle k \rangle}$$

Linearize:

$$\frac{d\Theta}{dt} = -\mu\Theta + \beta\Theta \sum \frac{k^2 P_k}{\langle k \rangle}$$

Outbreak when

$$\left(\frac{\beta\langle k^2 \rangle}{\langle k \rangle} - \mu\right)\Theta > 0$$

DBMF Threshold:

$$\beta_c \langle k \rangle = \mu / \langle k^2 \rangle$$

Homogeneous MF threshold:

$$\beta_c \langle k \rangle = \mu$$

For $P_k \sim k^{-\gamma}$ and $\gamma \leq 3$:

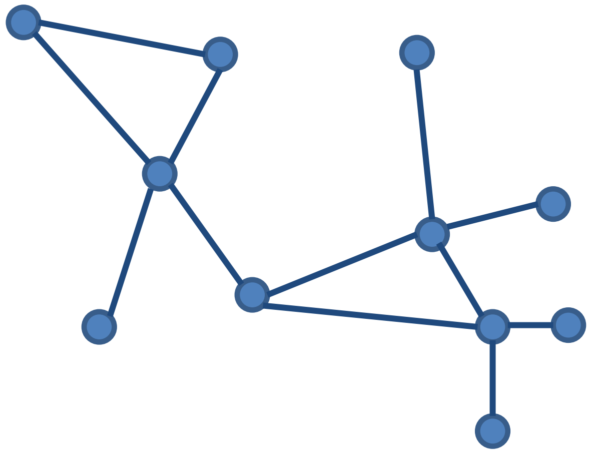
$$\langle k^2 \rangle \to \infty$$

Individual-based Mean Field (IBMF)
Quenched Mean Field (QMF)
N-Intertwined Mean Field Approach (NIMFA)
Microscopic Markov Chain Approach (MMCA)

Quenched network ☺

Ignoring dynamical correlations ☹

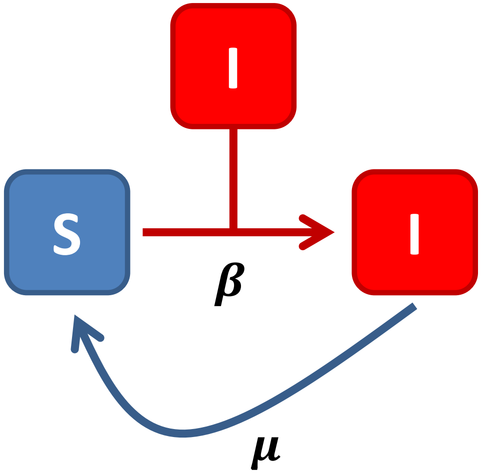
Individual-based mean field (IBMF)



Keep complete network A_{ij}.

Ignore dynamical correlations for process.

Probality that i is infected: ρ_i



***N* equations:**

$$\frac{d\rho_i}{dt} = -\mu\rho_i + \beta(1-\rho_i)\sum_j A_{ij}\rho_j$$

IBMF: SIS—Epidemic threshold

$$\frac{d\rho_i}{dt} = -\mu\rho_i + \beta(1-\rho_i)\sum_j A_{ij}\rho_j$$

Linearize:

$$\frac{d\boldsymbol{\rho}}{dt} = M\boldsymbol{\rho}$$

$$M_{ij} = -\mu\delta_{ij} + \beta A_{ij}$$

Outbreak when largest eigenvalue

of M is positive

λ_1 : largest eigenvalue of A

Epidemic threshold:

$$\beta_c = \mu/\lambda_1$$

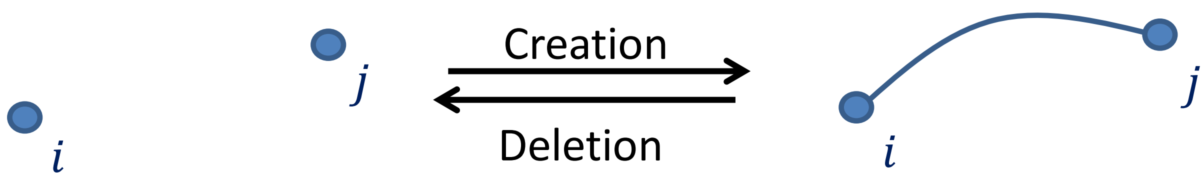
Individual-based Mean Field (IBMF) for temporal networks

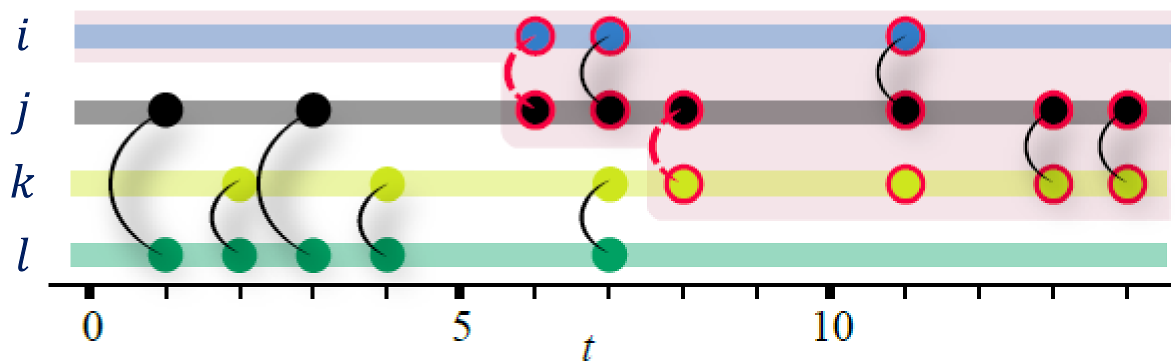
Temporal network ☺

Ignoring dynamical correlations of process ☹

Reference: Valdano *et al.*, arXiv:1406.4815

Temporal network contact dynamics





Reference: Vestergaard *et al.*, arXiv:1409.1805

Model of temporal network

N agents (nodes) i
$N(N-1)/2$ links (i,j)

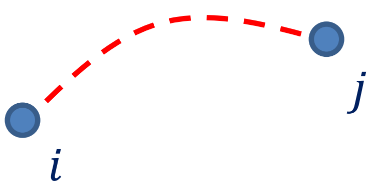
Contact = active link
No contact = inactive link

Stochastic algorithm: each time-step:

Contact deletion

for each contact (i,j):

Deletion



rate $= z\left(1+\tau_{(i,j)}\right)^{-1}$

Time since creation

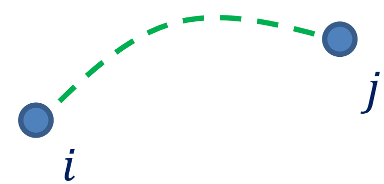
Contact creation

for each agent i:

Initiation

rate $= b(1+\tau_i)^{-1}$

Choice



proba. $\sim (1+\tau_i)^{-1}\left(1+\tau_{(i,j)}\right)^{-1}$

Time since a contact was gained/lost

Time since deletion

Distribution of (inter-)contact times

Distribution of contact durations:

$$p\left(\tau_{(i,j)}\right) \propto -\frac{dm_1\left(\tau_{(i,j)}\right)}{dt}$$

Number of active links of age $\tau_{(i,j)}$

Distribution of times between contacts for $\boldsymbol{i}$:

$$p(\Delta\tau_i) \propto -\frac{dn_0(\Delta\tau_i)}{dt}$$

Number of agents isolated since $\mathrm{t} - \tau_{(i,j)}$

Distribution of times between contacts of $(\boldsymbol{i}, \boldsymbol{j})$:

$$p\left(\Delta\tau_{(i,j)}\right) \propto -\frac{dm_0\left(\Delta\tau_{(i,j)}\right)}{dt}$$

Number of inactive links of age $\tau_{(i,j)}$

Mean-field master equations

Updating a contact (deletion):

$$dm_1(t,t') = -dt\, z(1+t-t')^{-1}\frac{m_1(t,t')}{M_1(t)}$$

$$dn_0(t,t') = \frac{\pi_{1,0}(t)}{M_1(t)}\delta(t-t')$$

$$dm_0(t,t') = r_-(t)\delta(t-t')$$

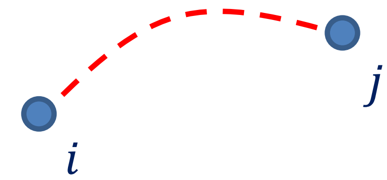
Updating an agent (creation):

$$dm_1(t,t') = r_+(t)\delta(t-t')$$

$$dn_0(t,t') = -[b+r_+(t)c(t)](1+t-t')^{-1}\frac{n_0(t,t')}{N}$$

$$dm_0(t,t') = d(t)(1+t-t')^{-1}\frac{m_0(t,t')}{M_0(t)}$$

Deletion

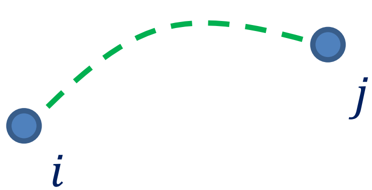


rate $= z\left(1+\tau_{(i,j)}\right)^{-1}$

Initiation

rate $= b(1+\tau_i)^{-1}$

Choice



proba. $\sim (1+\tau_i)^{-1}\left(1+\tau_{(i,j)}\right)^{-1}$

Continuous MF master equations

Number of active links of age $t - t'$:

$$\frac{dm_1(t,t')}{dt} = -z(1+t-t')^{-1}m_1(t,t') + Nr_+(t)\delta(t-t')$$

Number of agents isolated since t':

$$\frac{dn_0(t,t')}{dt} = -2b(1+t-t')^{-1}n_0(t,t') + \pi_{1,0}(t)\delta(t-t')$$

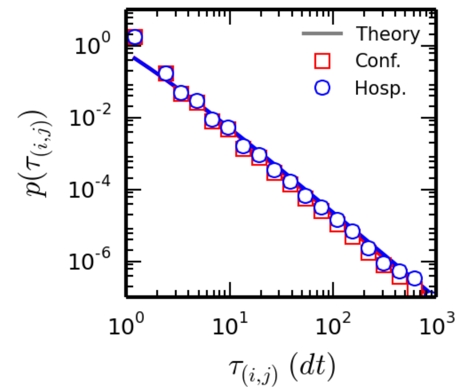
Number of inactive links of age $t - t'$:

$$\frac{dm_0(t,t')}{dt} = -\frac{Nd(t)}{M_0(t)}(1+t-t')^{-1}m_0(t,t') + M_1(t)r_-(t)\delta(t-t')$$

Distribution of (inter-)contact times

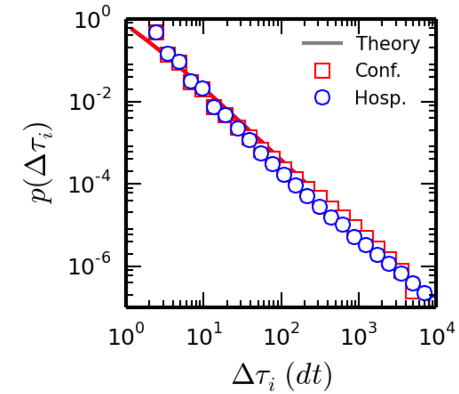
Distribution of contact durations:

$$p\left(\tau_{(i,j)}\right) = z\left(1 + \tau_{(i,j)}\right)^{-(z+1)}$$



Distribution of times between contacts for *i*:

$$p(\Delta\tau_i) = 2b(1 + \tau_i)^{-(2b+1)}$$



Distribution of times between contacts of $(\boldsymbol{i}, \boldsymbol{j})$:

$$p\left(\Delta\tau_{(i,j)}\right) = \alpha\left(1 + \tau_{(i,j)}\right)^{-(\alpha+1)}$$